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A Formula for the Separability Idempotent in the Tensor Square of a Field

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1. INTRODUCTION

We shall establish here an explicit formula for the separability idempotent ([1, 2]) in a ring of the form $E \otimes_k E$, where E is a separable finite extension of a ground field k. The formula is as follows.

Let $\mathbf{a}$ be a primitive element for E over k, and let $\mathbf{a}_1 = \mathbf{a}, \mathbf{a}_2, \mathbf{a}_3, ..., \mathbf{a}_d$ be its conjugates in the splitting field Ω of the minimal polynomial of $\mathbf{a}$. Let $V(\mathbf{a})$ be the matrix

$$V(\mathbf{a}) = \begin{pmatrix} 1 & \mathbf{a}_1 & \mathbf{a}_1^2 & \mathbf{a}_1^3 & \cdots & \mathbf{a}_1^{d-1} \\ 1 & \mathbf{a}_2 & \mathbf{a}_2^2 & \mathbf{a}_2^3 & \cdots & \mathbf{a}_2^{d-1} \\ & & & *** & & \\ 1 & \mathbf{a}_d & \mathbf{a}_d^2 & \mathbf{a}_d^3 & \cdots & \mathbf{a}_d^{d-1} \end{pmatrix}.$$

This defines a linear operator in Ω^d which is invertible by the assumed separability. The entries in ${}^tV(\mathbf{a}) \cdot V(\mathbf{a})$ are the power sums of the conjugates and so lie in k. Therefore the tensor Θ defined as

$$\Theta = \sum \{{}^tV(\mathbf{a})\, V(\mathbf{a})\}^{-1}{}_{ij}(\mathbf{a}^i \otimes \mathbf{a}^j) \tag{1}$$

is an element of $E \otimes_k E$. We will prove that it is the separability idempotent in $E \otimes_k E$. We will also show how the use of a primitive element may be avoided at the expense of further calculation.

2. PRELIMINARIES

We write $\mathbf{G}(K/F)$ for the group of automorphisms of a field $K \supset F$ fixing F pointwise, and **id** for the identity map. Any two maps $f, g: K \to K$ fixing F determine a map $f \times g$: $K \otimes_F K \to K$ defined by the relation $f \times g(x \otimes y) = f(x)\, g(y)$.

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We recall that if E is a separable finite extension of k then $E \otimes_k E$ is the algebra-direct-sum $E \otimes_k E = \bigoplus_j (E \otimes_k E)\, e_j$ of its minimal ideals $(E \otimes_k E)\, e_j$, each principal, generated by a uniquely determined indecomposable idempotent e_j; that these idempotents satisfy $e_j e_k = \delta_{jk} e_k$ and $\sum e_j = 1 \otimes 1$; that each minimal ideal is a field; and that all ideals in $E \otimes_k E$ have unique complements.

By the *support ideal* of a ring homomorphism $h: E \otimes_k E \to M$, M any extension of E, we mean the ideal complementary to $\ker(h)$. We say $T \in E \otimes_k E$ satisfies the *symmetry condition* if

$$(1 \otimes x - x \otimes 1)T = 0, \qquad \text{all } x \in E. \tag{2}$$

For any separable finite extension E of k there is precisely one idempotent in $E \otimes_k E$ satisfying (2), namely, the generator of the support ideal of $\mathbf{id} \times \mathbf{id}$. That idempotent is the *separability idempotent* in $E \otimes_k E$. It is indecomposable because the support ideal of $\mathbf{id} \times \mathbf{id}$ is a field.

3. Proof of the Formula

With matters and notation as in Section 1, we interprete Θ as an element of $\Omega \otimes_k \Omega$, and assert that, as such,

$$(\mathbf{id} \times \sigma)(\Theta) = \begin{cases} 1 & \text{if } \sigma \in \mathbf{G}(\Omega/E) \\ 0 & \text{if } \sigma \notin \mathbf{G}(\Omega/E). \end{cases} \tag{3}$$

To prove (3) we put

$$\vec{\mathbf{a}}_k = (1, \mathbf{a}_k, \mathbf{a}_k^2, \ldots, \mathbf{a}_k^{d-1}).$$

Then, if $\{\vec{\mathbf{b}}_k\}_1^d$ is that basis of Ω^d such that $(\vec{\mathbf{b}}_k)_j = \delta_{kj}$, we have

$${}^tV(\mathbf{a})\, \vec{\mathbf{b}}_k = \vec{\mathbf{a}}_k.$$

For $\sigma \in \mathbf{G}(\Omega/k)$ we have

$$\begin{aligned} (\mathbf{id} \times \sigma)(\Theta) &= \sum_{ij} \{{}^tV(\mathbf{a})\, V(\mathbf{a})\}^{-1}{}_{ij}(\mathbf{a}_1^i \cdot \sigma(\mathbf{a}_1^j)) \\ &= \langle V(\mathbf{a})^{-1}\, {}^tV(\mathbf{a})^{-1}\, \vec{\mathbf{a}}_1, \sigma(\vec{\mathbf{a}}_1)\rangle \\ &= \langle {}^tV(\mathbf{a})^{-1}\, \vec{\mathbf{a}}_1, {}^tV(\mathbf{a})^{-1}\, \sigma(\vec{\mathbf{a}}_1)\rangle \end{aligned}$$

in the natural scalar product in Ω^d. Now $\sigma(\vec{\mathbf{a}}_1) = \vec{\mathbf{a}}_1$ if $\sigma \in \mathbf{G}(\Omega/E)$, and $\sigma(\vec{\mathbf{a}}_1) = \vec{\mathbf{a}}_k$, $k \neq 1$, if $\sigma \notin \mathbf{G}(\Omega/E)$ because $E = k(\mathbf{a}_1)$. In the first case $(\mathbf{id} \times \sigma)(\Theta) = \langle {}^tV(\mathbf{a})^{-1}\, \vec{\mathbf{a}}_1, {}^tV(\mathbf{a})^{-1}\, \vec{\mathbf{a}}_1\rangle = \langle \vec{\mathbf{b}}_1, \vec{\mathbf{b}}_1\rangle = 1$. In the second case $(\mathbf{id} \times \sigma)(\Theta) = \langle \vec{\mathbf{b}}_1, \vec{\mathbf{b}}_k\rangle = 0$. This completes the proof of (3).

Since Ω is a Galois extension of k the set of indecomposable idempotents in $\Omega \otimes_k \Omega$ is the set $\{e_\sigma : \sigma \in \mathbf{G}(\Omega/k)\}$ where $e_\sigma = (\sigma \otimes \mathbf{id})\, e_1$, e_1 being the separability idempotent in $\Omega \otimes_k \Omega$. Thus $\Omega \otimes_k \Omega = \bigoplus_\sigma (\Omega \otimes_k \Omega)\, e_\sigma$, and it is immediate (from the fact that $(1 \otimes x - x \otimes 1)\, e_1 = 0$ for all $x \in \Omega$) that $(\Omega \otimes_k \Omega)\, e_\sigma = (\Omega \otimes_k 1)\, e_\sigma$. Reasoning from the way in which $\mathbf{G}(\Omega/k)$ acts upon $\{e_\sigma\}$, namely, $(\theta \otimes \eta)\, e_\sigma = e_{(\theta\sigma\eta^{-1})}$, one finds that the support ideal of $\mathbf{id} \times \sigma$ is $(\Omega \otimes_k \Omega)\, e_\sigma$. Writing $T \in \Omega \otimes_k \Omega$ as $T = \sum (T_\tau \otimes 1)\, e_\tau$, $T_\tau \in \Omega$, and applying $\mathbf{id} \times \sigma$ to both sides, we then have $T_\sigma = (\mathbf{id} \times \sigma)T$. In these terms (3) is the statement that $\Theta = \sum \{e_\sigma : \sigma \in \mathbf{G}(\Omega/E)\}$. In particular we thus know Θ to be idempotent. It remains only to check that Θ satisfies the symmetry condition (2) in $E \otimes_k E$. We do this by viewing $T = (1 \otimes x - x \otimes 1)\Theta$, $x \in E$, as an element of $\Omega \otimes_k \Omega$ and showing that all its "coordinates" $(\mathbf{id} \times \sigma)T = T_\sigma$ vanish. We have $(\mathbf{id} \times \sigma)T = (\sigma(x) - x) \cdot (\mathbf{id} \times \sigma)(\Theta)$. Now the first factor vanishes if $\sigma \in \mathbf{G}(\Omega/E)$, and by (3) the second factor vanishes if $\sigma \notin \mathbf{G}(\Omega/E)$. Therefore the product vanishes. This completes the proof.

We remark in closing that the use of a primitive element in the construction of the separability idempotent may be avoided as follows. If E is constructed by a series of adjunctions, say $E = k(r_1, r_2, \ldots, r_n)$, then the separability idempotent R_j in each of the intermediate algebras $k(r_j) \otimes_k k(r_j)$ is given by (1) with $\mathbf{a} = r_j$, and the product in $E \otimes_k E$ of these idempotents is the separability idempotent in $E \otimes_k E$. To see this it is sufficient to check that

$$(1 \otimes r_i - r_i \otimes 1) \left(\prod_j R_j \right) = 0, \qquad \text{all } i.$$

But $E \otimes_k E$ is commutative, and $(1 \otimes r_i - r_i \otimes 1)\, R_i = 0$ for all i, so $(1 \otimes r_i - r_i \otimes 1)(\prod_j R_j) = (\prod_{j \neq i} R_j)(1 \otimes r_i - r_i \otimes 1)\, R_i = 0$ for each i.

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References

1. F. DeMeyer and E. Ingraham, "Separable Algebras over Commutative Rings," Lecture Notes in Mathematics No. 181, Springer-Verlag, New York, 1971.
2. S. U. Chase, D. K. Harrison, and A. Rosenberg, Galois theory and cohomology of commutative rings, *Mem. Amer. Math. Soc.* **52** (1965).